9.4 Cirkel en raaklijn

Opgave 45:

$(x-3)^2+(y-2)^2=25$

$x^2-6x+9+y^2-4y+4=25$

$x^2+y^2-6x-4y-12=0$

Opgave 46:

$x^2+y^2+by+15=0$

$x^2+(y+\frac{1}{2}b)^2-\frac{1}{4}b^2+15=0$

$x^2+(y+\frac{1}{2}b)^2=\frac{1}{4}b^2-15$

het is een cirkel als geldt:

$\frac{1}{4}b^2-15\geq 0$

$\frac{1}{4}b^2\geq 15$

$b^2\geq 60$

$b\leq -\sqrt{60} \quad \vee \quad b\geq \sqrt{60}$

$b\leq -2\sqrt{15} \quad \vee \quad b\geq 2\sqrt{15}$

Opgave 47:

a. vul in $x=0$ en $y=0$ dan geldt: $0=0$

dus het punt $(0{,}0)$ ligt op de cirkel

b. $\begin{cases} 100+10a=0 \\ 16+36+4a+6b=0 \end{cases}$

$10a=-100$

$a=-10$

$52+4\cdot -10+6b=0$

$6b=-12$

$b=-2$

$x^2+y^2-10x-2y=0$

$(x-5)^2-25+(y-1)^2-1=0$

$(x-5)^2+(y-1)^2=26$

dus $M(5{,}1)$

c. $x^2+y^2+2bx+by=0$

$(x+b)^2-b^2+(y+\frac{1}{2}b)^2-\frac{1}{4}b^2=0$

$(x+b)^2+(y+\frac{1}{2}b)^2=1\frac{1}{4}b^2$

$1\frac{1}{4}b^2=5$

$b^2=4$

$b=2 \quad \vee \quad b=-2$ (vervalt)

$a=4$

dus $M(-2,-1)$

Opgave 48:

a. $x^2+y^2+ax-5y+6=0$

$(x+\frac{1}{2}a)^2-\frac{1}{4}a^2+(y-2\frac{1}{2})^2-6\frac{1}{4}+6=0$

$(x+\frac{1}{2}a)^2+(y-2\frac{1}{2})^2=\frac{1}{4}a^2+\frac{1}{4}$

voor iedere waarde van a geldt dat $\frac{1}{4}a^2+\frac{1}{4}>0$ dus heb je altijd een cirkel

b. $\frac{1}{4}a^2+\frac{1}{4}>25$

$\frac{1}{4}a^2>24\frac{3}{4}$

$a^2>99$

$a<-\sqrt{99} \quad \vee \quad a>\sqrt{99}$

$a<-3\sqrt{11} \quad \vee \quad a>3\sqrt{11}$

c. $M(-\frac{1}{2}a, 2\frac{1}{2})$

$-\frac{1}{2}a-2\cdot 2\frac{1}{2}+2=0$

$-\frac{1}{2}a-5+2=0$

$-\frac{1}{2}a=3$

$a=-6$

Opgave 49:

a. $x^2+y^2+4x-6y-24=0$

$(x+2)^2-4+(y-3)^2-9-24=0$

$(x+2)^2+(y-3)^2=37$

dus $M(-2,3)$ en $r=\sqrt{37}$

$AM=\sqrt{4^2+4^2}=\sqrt{32}<\sqrt{37}$ dus punt A ligt binnen de cirkel

b. $P(\lambda, 6-\lambda)$

$PM=\sqrt{(\lambda+2)^2+(6-\lambda-3)^2}$

$=\sqrt{(\lambda+2)^2+(3-\lambda)^2}$

$=\sqrt{\lambda^2+4\lambda+4+\lambda^2-6\lambda+9}$

$=\sqrt{2\lambda^2-2\lambda+13}$

$2\lambda^2-2\lambda+13<37$

$2\lambda^2-2\lambda-24<0$

$\lambda^2-\lambda-12<0$

$(\lambda+3)(\lambda-4)=0$

$\lambda=-3 \quad \vee \quad \lambda=4$

dus $-3<\lambda<4$

Opgave 50:

a. $x-2y=2$

$x=2+2y$

$(2+2y-3)^2+(y-2)^2=10$

$(2y-1)^2+(y-2)^2=10$

$4y^2-4y+1+y^2-4y+4=10$

$5y^2 - 8y - 5 = 0$

$y = \frac{8 \pm \sqrt{164}}{10} = \frac{8 \pm 2\sqrt{41}}{10}$

$y = \frac{4}{5} + \frac{1}{5}\sqrt{41} \quad \vee \quad y = \frac{4}{5} - \frac{1}{5}\sqrt{41}$

$y = \frac{4}{5} + \frac{1}{5}\sqrt{41}$ geeft $x = 2 + 2 \cdot (\frac{4}{5} + \frac{1}{5}\sqrt{41}) = 3\frac{3}{5} + \frac{2}{5}\sqrt{41}$

$y = \frac{4}{5} - \frac{1}{5}\sqrt{41}$ geeft $x = 2 + 2 \cdot (\frac{4}{5} - \frac{1}{5}\sqrt{41}) = 3\frac{3}{5} - \frac{2}{5}\sqrt{41}$

dus $(3\frac{3}{5} - \frac{2}{5}\sqrt{41}, \frac{4}{5} - \frac{1}{5}\sqrt{41})$ en $(3\frac{3}{5} + \frac{2}{5}\sqrt{41}, \frac{4}{5} + \frac{1}{5}\sqrt{41})$

b. $6x + 4y = 41$

$4y = -6x + 41$

$y = -1\frac{1}{2}x + 10\frac{1}{4}$

$x^2 + (-1\frac{1}{2}x + 10\frac{1}{4})^2 - 8x - 2(-1\frac{1}{2}x + 10\frac{1}{4}) + 10\frac{1}{2} = 0$

$x^2 + 2\frac{1}{4}x^2 - 30\frac{3}{4} + 105\frac{1}{16} - 8x + 3x - 20\frac{1}{2} + 10\frac{1}{2} = 0$

$3\frac{1}{4}x^2 - 35\frac{3}{4}x + 95\frac{1}{16} = 0$

$52x^2 - 572x + 1521 = 0$

$x = \frac{572 \pm \sqrt{10816}}{104} = \frac{572 \pm 104}{104}$

$x = 6\frac{1}{2} \quad \vee \quad x = 4\frac{1}{2}$

$y = \frac{1}{2} \quad\quad\quad y = 3\frac{1}{2}$

$(6\frac{1}{2}, \frac{1}{2})$ en $(4\frac{1}{2}, 3\frac{1}{2})$

Opgave 51:

a. $x^2 - 4x + y^2 = 0$

$(x-2)^2 - 4 + y^2 = 0$

$(x-2)^2 + y^2 = 4$

$M(2,0)$

b. $(0,0)$ invullen geeft: $0^2 - 4 \cdot 0 + 0^2 = 0$ dus klopt

c. $x^2 - 4x + (ax)^2 = 0$

$x^2 - 4x + a^2x^2 = 0$

$x(x - 4 + a^2x) = 0$

$x = 0 \quad \vee \quad x - 4 + a^2x = 0$

$(1 + a^2)x = 4$

$x = \frac{4}{1 + a^2}$

$y = 0 \quad \vee \quad y = a \cdot \frac{4}{1 + a^2} = \frac{4a}{1 + a^2}$

$(0,0)$ en $(\frac{4}{1 + a^2}, \frac{4a}{1 + a^2})$

d. $A = (\frac{4}{\frac{1}{4} + 1}, \frac{1}{\frac{1}{4} + 1}) = (3\frac{1}{5}, \frac{4}{5})$

e. $a = \frac{3}{4}$ dus $A = (\frac{4}{\frac{9}{16}+1}, \frac{3}{\frac{9}{16}+1}) = (2\frac{14}{25}, 1\frac{23}{25}0$

f. $x_P = \frac{4}{1+a^2} = \frac{50}{13}$

$50(1+a^2) = 52$

$50 + 50a^2 = 52$

$50a^2 = 2$

$a^2 = \frac{1}{15}$

$a = \frac{1}{5} \quad \vee \quad a = -\frac{1}{5}$

punt P onder de x-as dus $y_P < 0$ dus $a = -\frac{1}{5}$

dus $y = -\frac{1}{5}x$

Opgave 52:

a. $y = a(x+1)$

b. $x^2 + a^2(x+1)^2 = 1$

$x^2 + a^2(x^2 + 2x + 1) = 1$

$x^2 + a^2x^2 + 2a^2x + a^2 - 1 = 0$

$(1+a^2)x^2 + 2a^2x + a^2 - 1 = 0$

$D = (2a^2)^2 - 4 \cdot (1+a^2) \cdot (a^2 - 1)$

$D = 4a^4 - 4(-1 + a^4)$

$D = 4a^4 + 4 - 4a^4$

$D = 4$

$x = \frac{-2a^2 \pm \sqrt{4}}{2+2a^2} = \frac{-2a^2 \pm 2}{2+2a^2}$

$x = \frac{-2a^2 - 2}{2+2a^2} = -1 \quad \vee \quad x = \frac{-2a^2 + 2}{2+2a^2} = \frac{1-a^2}{a^2+1}$

$y = a \cdot (\frac{1-a^2}{a^2+1} + 1) = a \cdot (\frac{1-a^2}{a^2+1} + \frac{a^2+1}{a^2+1}) = a \cdot (\frac{2}{a^2+1}) = \frac{2a}{a^2+1}$

c. $a = \frac{1}{2}$ geeft $P(\frac{3}{5}, \frac{4}{5})$

$x = \frac{3}{5} \quad y = \frac{4}{5} \quad r = 1$

$a = \frac{3}{5} \quad b = \frac{4}{5} \quad c = 1$

dus $(3,4,5)$

$a = \frac{1}{4}$ geeft $P(\frac{15}{17}, \frac{8}{17})$

$a = \frac{15}{17} \quad b = \frac{8}{17} \quad c = 1$

dus $(15,8,17)$

d. $(5,12,13)$ dus $x = \frac{5}{13}$ en $y = \frac{12}{13}$

$\frac{1-a^2}{a^2+1} = \frac{5}{13}$

$5(a^2+1) = 13(1-a^2)$

$5a^2 + 5 = 13 - 13a^2$

$18a^2 = 8$

$a^2 = \frac{4}{9}$

$a = \frac{2}{3} \quad \vee \quad a = -\frac{2}{3}$ (vervalt)

e. $(708, 13915, 13933)$ dus $x = \frac{708}{13933}$

$\frac{1-a^2}{a^2+1} = \frac{708}{13933}$

$708(a^2+1) = 13933(1-a^2)$

$708a^2 = 708 = 13933 - 13933a^2$

$14641a^2 = 13225$

$a^2 = \frac{13225}{14641}$

$a = \frac{115}{121} \quad \vee \quad a = -\frac{115}{121}$ (vervalt)

Opgave 53:

a. $y = 2x + 5$

$x^2 + (2x+5)^2 = 10$

$x^2 + 4x^2 + 20x + 25 = 10$

$5x^2 + 20x + 15 = 0$

$x^2 + 4x + 3 = 0$

$(x+3)(x+1) = 0$

$x = -3 \quad \vee \quad x = -1$

$y = 3 \qquad\quad y = -1$

dus $(-3,-1)$ en $(-1,3)$

b. $y = 2x + 5\sqrt{2}$

$x^2 + (2x + 5\sqrt{2})^2 = 10$

$x^2 + 4x^2 + 20x\sqrt{2} + 50 = 10$

$5x^2 + 20x\sqrt{2} + 40 = 0$

$x^2 + 4x\sqrt{2} + 8 = 0$

$D = (4\sqrt{2})^2 - 4 \cdot 1 \cdot 8 = 32 - 32 = 0$

dus er is één oplossing, dus raakt de lijn de cirkel

Opgave 54:

a. $rc_m = -\frac{1}{3}$ en $M(0,0)$

lijn m: $y = -\frac{1}{3}x$

$x^2 + (-\frac{1}{3}x)^2 = 10$

$x^2 + \frac{1}{9}x^2 = 10$

$1\frac{1}{9}x^2 = 10$

$x^2 = 9$

$x = 3 \quad \vee \quad x = -3$

$y = -1 \qquad y = 1$

$A(3,-1) \quad B(-3,1)$

door A: $y + 1 = 3(x-3)$ dus $y = 3x - 10$

door B: $y - 1 = 3(x+3)$ dus $y = 3x + 10$

b. lijn door $(10,0)$: $y = a(x-10)$

$x^2 + a^2(x-10)^2 = 10$

$x^2 + a^2(x^2 - 20x + 100) = 10$

$x^2 + a^2x^2 - 20a^2x + 100a^2 - 10 = 0$

$(1+a^2)x^2 - 20a^2x + 100a^2 - 10 = 0$

$D = (-20a^2)^2 - 4\cdot(1+a^2)(100a^2 - 10)$

$= 400a^4 - 4(100a^4 + 90a^2 - 10)$

$= 400a^4 - 400a^4 - 360a^2 + 40$

$= -360a^2 + 40 = 0$

$-360a^2 = -40$

$a^2 = \frac{1}{9}$

$a = \frac{1}{3} \quad \vee \quad a = -\frac{1}{3}$

$y = \frac{1}{3}(x-10)$ en $y = -\frac{1}{3}(x-10)$

Opgave 55:

Het punt $A(x_A, y_A)$ ligt op de cirkel $x^2+y^2 = r^2$

De raaklijn k aan de cirkel staat loodrecht op de straal OA.

$\underline{r}_{OA} = \begin{pmatrix} x_A \\ y_A \end{pmatrix} = \underline{n}_k$

dus $x_A \cdot x + y_A \cdot y = c$ door het punt $A(x_A, y_A)$

dus $x_A{}^2 + y_A{}^2 = c$

omdat punt A op de cirkel ligt geldt: $x_A{}^2 + y_A{}^2 = r^2$

dus $c = r^2$

dus de raaklijn k is: $x_A \cdot x + y_A \cdot y = r^2$

Opgave 56:

a. $2x + 3y = 13$

b. l: $y = 1\frac{1}{2}x + b$

$1\frac{1}{2}x - y + b = 0$

$d(M,l) = \dfrac{|0 - 0 + b|}{\sqrt{(1\frac{1}{2})^2 + (-1)^2}} = \sqrt{13}$

$\dfrac{|b|}{\sqrt{3\frac{1}{4}}} = \sqrt{13}$

$|b| = \sqrt{42\frac{1}{4}}$

$|b| = 6\frac{1}{2}$

$b = 6\frac{1}{2} \quad \vee \quad b = -6\frac{1}{2}$

l_1: $y = 1\frac{1}{2}x - 6\frac{1}{2}$ en l_2: $y = 1\frac{1}{2}x + 6\frac{1}{2}$

c. lijn l door $(4\frac{1}{3}, 0)$

$y = a(x - 4\frac{1}{3})$

$y = ax - 4\frac{1}{3}a$

$ax - y - 4\frac{1}{3}a = 0$

$d(M,l) = \dfrac{|0 - 0 - 4\frac{1}{3}a|}{\sqrt{a^2 + (-1)^2}} = \sqrt{13}$

$\dfrac{|-4\frac{1}{3}a|}{\sqrt{a^2 + 1}} = \sqrt{13}$

$|4\frac{1}{3}a| = \sqrt{13(a^2 + 1)}$

$18\frac{7}{9}a^2 = 13(a^2 + 1)$

$18\frac{7}{9}a^2 = 13a^2 + 13$

$5\frac{7}{9}a^2 = 13$

$a^2 = 2\frac{1}{4}$

$a = 1\frac{1}{2} \quad \vee \quad a = -1\frac{1}{2}$

$l_1: \quad y = 1\frac{1}{2}(x - 4\frac{1}{3})$ en $l_2: \quad y = -1\frac{1}{2}(x - 4\frac{1}{3})$

d. lijn l door $(1,5)$

$y - 5 = a(x - 1)$

$y - 5 = ax - a$

$ax - y + 5 - a = 0$

$d(M,l) = \dfrac{|0 - 0 + 5 - a|}{\sqrt{a^2 + (-1)^2}} = \sqrt{13}$

$\dfrac{|5 - a|}{\sqrt{a^2 + 1}} = \sqrt{13}$

$|5 - a| = \sqrt{13(a^2 + 1)}$

$(5 - a)^2 = 13(a^2 + 1)$

$a^2 - 10a + 25 = 13a^2 + 13$

$12a^2 + 10a - 12 = 0$

$a = \dfrac{-10 \pm \sqrt{676}}{24} = \dfrac{-10 \pm 26}{24}$

$a = \frac{2}{3} \quad \vee \quad a = -1\frac{1}{2}$

$l_1: \quad y = \frac{2}{3}(x - 1)$ en $l_2: \quad y = -1\frac{1}{2}(x - 1)$

Opgave 57:

a. lijn l door $(-4,-1)$

$y + 1 = a(x + 4)$

$y + 1 = ax + 4a$

$ax - y + 4a - 1 = 0$

$d(M,l) = \dfrac{|0 - 0 + 4a - 1|}{\sqrt{a^2 + (-1)^2}} = \sqrt{17}$

$$\frac{|4a-1|}{\sqrt{a^2+1}}=\sqrt{17}$$

$|4a-1|=\sqrt{17(a^2+1)}$

$(4a-1)^2=17(a^2+1)$

$16a^2-8a+1=17a^2+17$

$a^2+8a+16=0$

$(a+4)^2=0$

$a=-4$

$l: \quad y+1=-4(x+4)$

dus $l: \quad 4x+y=-17$

b. $l: \quad 4x-y=3$

$-y=-4x+3$

$y=4x-3$

$rc_l=4$

$rc_m=-\frac{1}{4}$

$m: \quad y=-\frac{1}{4}x+b$

$\frac{1}{4}x+y-b=0$

$$d(M,m)=\frac{|0-0-b|}{\sqrt{(\frac{1}{4})^2+1^2}}=\sqrt{17}$$

$$\frac{|-b|}{\sqrt{1\frac{1}{16}}}=\sqrt{17}$$

$|-b|=\sqrt{\frac{289}{16}}$

$b^2=\frac{289}{16}$

$b=\frac{17}{4} \quad \vee \quad b=-\frac{17}{4}$

$m_1: \quad y=-\frac{1}{4}x+4\frac{1}{4}$ en $m_2: \quad y=-\frac{1}{4}x-4\frac{1}{4}$

c. lijn l door $(0,17)$

$y=ax+17$

$ax-y+17=0$

$$d(M,l)=\frac{|0-0+17|}{\sqrt{a^2+(-1)^2}}=\sqrt{17}$$

$$\frac{|17|}{\sqrt{a^2+1}}=\sqrt{17}$$

$|17|=\sqrt{17(a^2+1)}$

$289=17(a^2+1)$

$289=17a^2+17$

$17a^2=272$

$a^2=16$

$a=4 \quad \vee \quad a=-4$

$l_1: \quad y = 4x + 17$ en $l_2: \quad y = -4x + 17$