14.3 Het differentiaalquotiënt bij krommen

Opgave 30:

a. $\frac{dx}{dt} = x'(t) = 2t - 4$

$\frac{dy}{dt} = y'(t) = 2$

b. punt T is het punt op de parabool waarvan de x-coördinaat minimaal is, dus $x'(t) = 0$, dus $\frac{dx}{dt} = 0$

c. $2t - 4 = 0$

$2t = 4$

$t = 2$

$T(-4,-2)$

Opgave 31:

a. evenwijdig aan de x-as dus: $\frac{dy}{dt} = 0$

$\frac{dy}{dt} = 3t^2 - 3 = 0$

$3t^2 = 3$

$t^2 = 1$

$t = 1 \quad \vee \quad t = -1$

$(-3,-2)$ en $(-3,2)$

evenwijdig aan de y-as dus: $\frac{dx}{dt} = 0$

$\frac{dx}{dt} = 2t = 0$

$t = 0$

$(-4,0)$

b. $x = -1$ dus $t^2 - 4 = -1$

$t^2 = 3$

$t = \sqrt{3} \quad \vee \quad t = -\sqrt{3}$

$y = 0$ dus $t^3 - 3t = 0$

$t(t^2 - 3) = 0$

$t = 0 \quad \vee \quad t^2 = 3$

$t = 0 \quad \vee \quad t = \sqrt{3} \quad \vee \quad t = -\sqrt{3}$

dus K is voor $t = -\sqrt{3}$ en voor $t = \sqrt{3}$ in het punt $(-1,0)$

$\frac{dy}{dx} = \frac{3t^2 - 3}{2t}$

voor $t = -\sqrt{3}$ geldt: $\frac{dy}{dx} = \frac{6}{-2\sqrt{3}} = -\sqrt{3}$

$y = -\sqrt{3} \cdot x + b$ door $(-1,0)$

$0 = \sqrt{3} + b$

$b = -\sqrt{3}$

$y = -\sqrt{3} \cdot x - \sqrt{3}$

voor $t = \sqrt{3}$ geldt: $\frac{dy}{dx} = \frac{6}{2\sqrt{3}} = \sqrt{3}$

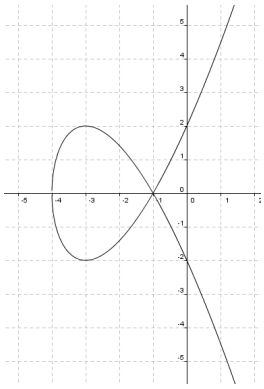
$y = \sqrt{3} \cdot x + b$ door $(-1,0)$

$0 = -\sqrt{3} + b$

$b = \sqrt{3}$

$y = \sqrt{3} \cdot x + \sqrt{3}$

de grafiek snijdt zichzelf niet loodrecht, want $\sqrt{3} \cdot -\sqrt{3} \neq -1$

c. 

Opgave 32:

a. snijpunten x-as: $y = 4\sin t = 0$

$t = 0 \quad \vee \quad t = \pi \quad \vee \quad t = 2\pi$

(1,0) en (–3,0)

snijpunten y-as: $x = 2\cos t - 1 = 0$

$2\cos t = 1$

$\cos t = \frac{1}{2}$

$t = \frac{1}{3}\pi \quad \vee \quad t = \frac{5}{3}\pi$

$(0,2\sqrt{3})$ en $(0,-2\sqrt{3})$

b. $-1 \leq \cos t \leq 1$

$-2 \leq 2\cos t \leq 2$

$-3 \leq 2\cos t - 1 \leq 1$ dus $-3 \leq x \leq 1$

$-1 \leq \sin t \leq 1$

$-4 \leq 4\sin t \leq 4$ dus $-4 \leq y \leq 4$

c. evenwijdig x-as: $\frac{dy}{dt} = 4\cos t = 0$

$t = \frac{1}{2}\pi \quad \vee \quad t = 1\frac{1}{2}\pi$

(–1,4) en (–1,–4)

evenwijdig y-as: $\frac{dx}{dt} = -2\sin t = 0$

$t = 0 \quad \vee \quad t = \pi \quad \vee \quad t = 2\pi$

(1,0) en (–3,0)

d. $\frac{dy}{dx} = \frac{4\cos t}{-2\sin t} = 2$

$4\cos t = -4\sin t$

$\cos t = -\sin t$

$\cos t = \cos(t + \frac{1}{2}\pi)$

$t = t + \frac{1}{2}\pi + k \cdot 2\pi \quad \vee \quad t = -t - \frac{1}{2}\pi + k \cdot 2\pi$

k.n. $2t = -\frac{1}{2}\pi + k \cdot 2\pi$

$t = -\frac{1}{4}\pi + k \cdot \pi$

$t = \frac{3}{4}\pi \quad \vee \quad t = 1\frac{3}{4}\pi$

dus $(-\sqrt{2}-1, 2\sqrt{2})$ en $(\sqrt{2}-1, -2\sqrt{2})$

e. $2\cos t = x+1$
$4\cos t = 2x+2$
$16\cos^2 t + 16\sin^2 t = 16$
$(4\cos t)^2 + (4\sin t)^2 = 16$
$(2x+2)^2 + y^2 = 16$
$4(x+1)^2 + y^2 = 16$

Opgave 33:

a. $t > 0$
$\frac{dx}{dt} = 2t - 4 = 0$
$2t = 4$
$t = 2$
$x = -4$ dus $x \geq -4$
$\frac{dy}{dt} = 1 \cdot \ln t + t \cdot \frac{1}{t} = \ln t + 1 = 0$
$\ln t = -1$
$t = e^{-1} = \frac{1}{e}$
$y = -\frac{1}{e}$ dus $y \geq -\frac{1}{e}$

b. evenwijdig x-as: $t = \frac{1}{e}$ dus $(\frac{1}{e^2} - \frac{4}{e}, -\frac{1}{e})$
evenwijdig y-as: $t = 2$ dus $(-4, 2\ln 2)$

c. $y = t \ln t = 0$
$t = 0 \quad \vee \quad \ln t = 0$
k.n. $t = 1$
$(-3,0)$
$\frac{dy}{dx} = -\frac{1}{2}$
$y = -\frac{1}{2}x + b$ door $(-3,0)$
$0 = 1\frac{1}{2} + b$
$b = -1\frac{1}{2}$
$y = -\frac{1}{2}x - 1\frac{1}{2}$

Opgave 34:

a. evenwijdig x-as: $\frac{dy}{dt} = \frac{1}{2}t - 1$
$-\frac{1}{2}t = -1$
$t = 2$
$(0,-1)$
evenwijdig y-as: $\frac{dx}{dt} = \frac{1}{2}t = 0$
$t = 0$
$(-1,0)$

b. $y = x$ dus $\frac{1}{4}t^2 - 1 = \frac{1}{4}t^2 - t$
$t = 1$
$(-\frac{3}{4}, -\frac{3}{4})$

voor $t=1$ is $\frac{dy}{dx}=\frac{-\frac{1}{2}}{\frac{1}{2}}=-1$

$y=-x+b$ door $(-\frac{3}{4},-\frac{3}{4})$

$-\frac{3}{4}=\frac{3}{4}+b$

$b=-1\frac{1}{2}$

$y=-x-1\frac{1}{2}$

c. snijpunt x-as: $\frac{1}{4}t^2-t=0$

$t(\frac{1}{4}t-1)=0$

$t=0 \quad \vee \quad \frac{1}{4}t=1$

$t=0 \quad \vee \quad t=4$

$(-1,0)$ en $(3,0)$

dus $B(3,0)$

snijpunt y-as: $\frac{1}{4}t^2-1=0$

$\frac{1}{4}t^2=1$

$t^2=4$

$t=2 \quad \vee \quad t=-2$

$(0,3)$ en $(0,-1)$

dus $C(0,3)$

in $(3,0)$ geldt: $\frac{dy}{dx}=\frac{1}{2}$ dus $y=\frac{1}{2}x+b$ door $(3,0)$

$0=1\frac{1}{2}+b$

$b=-1\frac{1}{2}$

$y=\frac{1}{2}x-1\frac{1}{2}$

in $(0,3)$ geldt: $\frac{dy}{dx}=2$ dus $y=2x+b$ door $(0,3)$

$y=2x+3$

$\frac{1}{2}x-1\frac{1}{2}=2x+3$

$-1\frac{1}{2}x=4\frac{1}{2}$

$x=-3$

$y=-3$

$D(-3,-3)$

d. $\frac{1}{4}t^2=x+1$

$y=x+1-t$ dus $t=x-y+1$

$x=\frac{1}{4}t^2-1$

$4x=t^2-4$

$4x=(x-y+1)^2-4$

Opgave 35:

a. de kettingregel

b. $(f(t))^2+4\cdot(g(t))^2=1$

$2f(t)\cdot f'(t)+8g(t)\cdot g'(t)=0$

$f(t)\cdot f'(t)+4g(t)\cdot g'(t)=0$

$x \cdot \frac{dx}{dt} + 4y \cdot \frac{dy}{dt} = 0$

c. $4y \cdot \frac{dy}{dt} = -x \cdot \frac{dx}{dt}$

$\frac{dy}{dx} = -\frac{x}{4y}$

Opgave 36:

a. $\frac{dy}{dx} = \frac{x^2 + 2y}{-2x + 2y} = 0$

$x^2 + 2y = 0$

$2y = -x^2$

$y = -\frac{1}{2}x^2$

$x^3 + 6x \cdot -\frac{1}{2}x^2 - 3(-\frac{1}{2}x^2)^2 = 0$

$x^3 - 3x^3 - \frac{3}{4}x^4 = 0$

$-2x^3 - \frac{3}{4}x^4 = 0$

$x^3(-2 - \frac{3}{4}x) = 0$

$x = 0 \quad \vee \quad -\frac{3}{4}x = 2$

k.n. $\quad x = -\frac{8}{3}$

$y = -\frac{32}{9}$

$B(-2\frac{2}{3}, -3\frac{5}{9})$

b. $\frac{dy}{dx} = \frac{x^2 + 2y}{-2x + 2y} = 1$

$x^2 + 2y = -2x + 2y$

$x^2 + 2x = 0$

$x(x + 2) = 0$

$x = 0 \quad \vee \quad x = -2$

$y = 0 \qquad -8 - 12y - 3y^2 = 0$

vervalt $\qquad y = \frac{12 \pm \sqrt{48}}{-6} = -2 \pm \frac{2}{3}\sqrt{3}$

$y = x + b$ door $(-2, -2 + \frac{2}{3}\sqrt{3})$

$-2 + \frac{2}{3}\sqrt{3} = -2 + b$

$b = \frac{2}{3}\sqrt{3}$

$y = x + \frac{2}{3}\sqrt{3}$

$y = x + b$ door $(-2, -2 - \frac{2}{3}\sqrt{3})$

$-2 - \frac{2}{3}\sqrt{3} = -2 + b$

$b = -\frac{2}{3}\sqrt{3}$

$y = x - \frac{2}{3}\sqrt{3}$

Opgave 37:

a. $x^2 + xy + y^2 = 9$ snijden met $x = 3$

$9 + 3y + y^2 = 9$

$y(y + 3) = 0$

$y = 0 \quad \vee \quad y = -3$

$(3,0)$ en $(3,-3)$

$2xdx + ydx + xdy + 2ydy = 0$

$(x+2y)dy = (-2x-y)dx$

$\frac{dy}{dx} = \frac{-2x-y}{x+2y}$

$\left[\frac{dy}{dx}\right]_{(3,0)} = -2$ $\qquad$ $\left[\frac{dy}{dx}\right]_{(3,-3)} = 1$

$y = -2x + b$ door $(3,0)$ $\qquad$ $y = x + b$ door $(3,-3)$

$0 = -6 + b$ $\qquad$ $-3 = 3 + b$

$b = 6$ $\qquad$ $b = -6$

$y = -2x + 6$ $\qquad$ $y = x - 6$

b. horizontale raaklijn: $-2x - y = 0 \quad \wedge \quad x + 2y \neq 0$

$y = -2x$

$x^2 - 2x^2 + 4x^2 = 9$

$3x^2 = 9$

$x^2 = 3$

$x = \sqrt{3} \quad \vee \quad x = -\sqrt{3}$

$y = -2\sqrt{3} \qquad y = 2\sqrt{3}$

dus $(\sqrt{3},-2\sqrt{3})$ en $(-\sqrt{3},2\sqrt{3})$

verticale raaklijn: $x + 2y = 0 \quad \wedge \quad -2x - y \neq 0$

$x = -2y$

$4y^2 - 2y^2 + y^2 = 9$

$3y^2 = 9$

$y^2 = 3$

$y = \sqrt{3} \quad \vee \quad y = -\sqrt{3}$

$x = -2\sqrt{3} \qquad x = 2\sqrt{3}$

dus $(-2\sqrt{3},\sqrt{3})$ en $(2\sqrt{3},-\sqrt{3})$

Opgave 38:

a. $K: \quad 9x^2 - 3y^2 = y^3$

$18xdx - 6ydy = 3y^2 dy$

$18xdx = (3y^2 + 6y)dy$

$\frac{dy}{dx} = \frac{18x}{3y^2 + 6y} = \frac{6x}{y^2 + 2y}$

horizontale raaklijn: $6x = 0 \quad \wedge \quad y^2 + 2y \neq 0$

$x = 0$

$-3y^2 = y^3$

$y^3 + 3y^2 = 0$

$y^2(y+3) = 0$

$y = -3 \qquad \vee \quad y = 0$ (vervalt)

dus $(0,-3)$

verticale raaklijn: $y^2 + 2y = 0 \quad \wedge \quad 6x \neq 0$

$y(y+2) = 0$

$y = 0 \quad \vee \quad y = -2$

$9x^2 = 0$ $\quad$ $9x^2 - 12 = -8$

$x = 0$ $\quad$ $9x^2 = 4$

$(0,0)$ $\quad$ $x^2 = \frac{4}{9}$

vervalt $\quad$ $x = \frac{2}{3} \quad \vee \quad x = -\frac{2}{3}$

$(\frac{2}{3},-2)$ en $(-\frac{2}{3},-2)$

b. $\frac{dy}{dx} = \frac{6x}{y^2 + 2y} = 1$

$6x = y^2 + 2y$

$3x = \frac{1}{2}y^2 + y$

$9x^2 = \frac{1}{4}y^4 + y^3 + y^2$

$\frac{1}{4}y^4 + y^3 + y^2 - 3y^2 = y^3$

$\frac{1}{4}y^4 - 2y^2 = 0$

$\frac{1}{4}y^2(y^2 - 8) = 0$

$y = 0 \quad \vee \quad y^2 = 8$

$x = 0$ $\quad$ $y = \sqrt{8} \quad \vee \quad y = -\sqrt{8}$

vervalt $\quad$ $6x = 8 + 2\sqrt{8} \quad \vee \quad 6x = 8 - 2\sqrt{8}$

$x = \frac{4}{3} + \frac{2}{3}\sqrt{2}$ $\quad$ $x = \frac{4}{3} - \frac{2}{3}\sqrt{2}$

dus $(\frac{4}{3} + \frac{2}{3}\sqrt{2}, 2\sqrt{2})$ en $(\frac{4}{3} - \frac{2}{3}\sqrt{2}, -2\sqrt{2})$

Opgave 39:

a. $x^4 - 4x^2 + 4 \cdot (px)^2 = 0$

$x^4 - 4x^2 + 4p^2x^2 = 0$

$x^2(x^2 - 4 + 4p^2) = 0$

$x = 0 \quad \vee \quad x^2 - 4 + 4p^2 = 0$ heeft 2 oplossingen

$D = 0 - 4 \cdot 1 \cdot (-4 + 4p^2) > 0$

$16 - 16p^2 > 0$

$-16p^2 > -16$

$p^2 < 1$

$-1 < p < 1$

b. $x^4 - 4x^2 + 4 \cdot \frac{1}{4}x^2 = 0$

$x^4 - 3x^2 = 0$

$x^2(x^2 - 3) = 0$

$x = 0 \quad \vee \quad x^2 = 3$

$x = \sqrt{3} \quad \vee \quad x = -\sqrt{3}$

$A(\sqrt{3}, \frac{1}{2}\sqrt{3})$

$4x^3dx - 8xdx + 8ydy = 0$

$8ydy = (8x - 4x^3)dx$

$\frac{dy}{dx} = \frac{8x - 4x^3}{8y} = \frac{2x - x^3}{2y}$

$\left[\frac{dy}{dx}\right]_{(\sqrt{3},\frac{1}{2}\sqrt{3})} = -1$

k: $y = -x + b$ door $(\sqrt{3}, \frac{1}{2}\sqrt{3})$

$12\sqrt{3} = -\sqrt{3} + b$

$b = 1\frac{1}{2}\sqrt{3}$

$y = -x + 1\frac{1}{2}\sqrt{3}$

c. horizontale raaklijn: $2x - x^3 = 0 \quad \wedge \quad 2y \neq 0$

$x(2 - x^2) = 0$

$x = 0 \quad \vee \quad x^2 = 2$

$y = 0 \qquad x = \sqrt{2} \quad \vee \quad x = -\sqrt{2}$

vervalt $\qquad 4 - 8 + 4y^2 = 0 \quad \vee \quad 4 - 8 + 4y^2 = 0$

$4y^2 = 4$

$y^2 = 1$

$y = 1 \quad \vee \quad y = -1$

$(\sqrt{2},1) \quad (\sqrt{2},-1) \quad (-\sqrt{2},1) \quad (-\sqrt{2},-1)$

verticale raaklijn: $2y = 0 \quad \wedge \quad 2x - x^3 \neq 0$

$y = 0$

$x^4 - 4x^2 = 0$

$x^2(x^2 - 4) = 0$

$x = 0 \quad \vee \quad x^2 = 4$

vervalt $\qquad x = 2 \quad \vee \quad x = -2$

$(2,0) \quad (-2,0)$

d. $\frac{dy}{dx} = \frac{2x - x^3}{2y} = 1$

$2y = 2x - x^3$

$4y^2 = 4x^2 - 4x^4 + x^6$

$x^4 - 4x^2 + 4x^2 - 4x^4 + x^6 = 0$

$x^6 - 3x^4 = 0$

$x^4(x^2 - 3) = 0$

$x = 0 \quad \vee \quad x^2 = 3$

vervalt $\qquad x = \sqrt{3} \quad \vee \quad x = -\sqrt{3}$

$2y = 2\sqrt{3} - 3\sqrt{3} = -\sqrt{3} \quad \vee \quad 2y = -2\sqrt{3} + 3\sqrt{3} = \sqrt{3}$

$y = -\frac{1}{2}\sqrt{3} \quad \vee \quad y = \frac{1}{2}\sqrt{3}$

$y = x + q$ door $(\sqrt{3}, -\frac{1}{2}\sqrt{3})$ $\qquad$ $y = x + q$ door $(-\sqrt{3}, \frac{1}{2}\sqrt{3})$

$-\frac{1}{2}\sqrt{3} = \sqrt{3} + q$ $\qquad$ $\frac{1}{2}\sqrt{3} = -\sqrt{3} + q$

$q = -1\frac{1}{2}\sqrt{3}$ $\qquad$ $q = 1\frac{1}{2}\sqrt{3}$

dus $q < -1\frac{1}{2}\sqrt{3} \quad \vee \quad q > 1\frac{1}{2}\sqrt{3}$

Opgave 40:

a. $y^2 - 2x - 2y + 3 = 0$

$(y-1)^2 - 1 - 2x + 3 = 0$

$(y-1)^2 = 2x - 2$

$(5-1)(y-1) = 9 - 1 + x - 1$

$4(y-1) = 7 + x$

$4y - 4 = 7 + x$

$x - 4y = 11$

$2ydy - 2dx - 2dy = 0$

$(2y-2)dy = 2dx$

$\frac{dy}{dx} = \frac{2}{2y-2}$

$\left[\frac{dy}{dx}\right]_{(9,5)} = \frac{1}{4}$

$y = \frac{1}{4}x + b$ door (9,5)

$y - 5 = \frac{1}{4}(x-9)$

$y = \frac{1}{4}x + 2\frac{3}{4}$

b. $3x^2 + 5y^2 - 18x - 20y - 48 = 0$

$3(x^2 - 6x) + 5(y^2 - 4y) = 48$

$3((x-3)^2 - 9) + 5((y-2)^2 - 4) = 48$

$3(x-3)^2 - 27 + 5(y-2)^2 - 20 = 48$

$3(x-3)^2 + 5(y-2)^2 = 95$

$3(8-3)(x-3) + 5(4-2)(y-2) = 95$

$15(x-3) + 10(y-2) = 95$

$15x - 45 + 10y - 20 = 95$

$15x + 10y = 160$

$3x + 2y = 32$

$6xdx + 10ydy - 18dx - 20dy = 0$

$(10y-20)dy = (18-6x)dx$

$\frac{dy}{dx} = \frac{18-6x}{10y-20}$

$\left[\frac{dy}{dx}\right]_{(8,4)} = -1\frac{1}{2}$

$y = -1\frac{1}{2}x + b$ door (8,4)

$y - 4 = -1\frac{1}{2}(x-8)$

$y = -1\frac{1}{2}x + 16$

c. $3x^2 - 5y^2 - 12x + 30y - 205 = 0$

$3(x^2 - 4x) - 5(y^2 - 6y) = 205$

$3((x-2)^2 - 4) - 5((y-3)^2 - 9) = 205$

$3(x-2)^2 - 12 - 5(y-3)^2 + 45 = 205$

$3(x-2)^2 - 5(y-3)^2 = 172$

$3(10-2)(x-2) - 5(5-3)(y-3) = 172$

$24(x-2) - 10(y-3) = 172$

$24x - 48 - 10y + 30 = 172$

$24x - 10y = 190$

$12x - 5y = 95$

$6xdx - 10ydy - 12dx + 30dy = 0$

$(30-10y)dy = (12-6x)dx$

$\frac{dy}{dx} = \frac{12-6x}{30-10y}$

$\left[\frac{dy}{dx}\right]_{(10,5)} = 2\frac{2}{5}$

$y = 2\frac{2}{5}x + b$ door (10,5)

$y - 5 = 2\frac{2}{5}(x-10)$

$y = 2\frac{2}{5}x - 19$