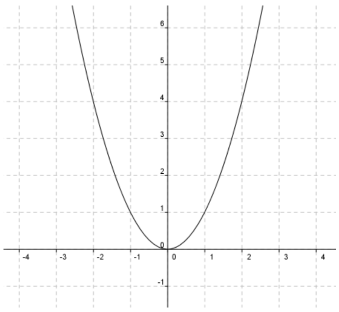
Hoofdstuk 16 Limieten

16.1 Continuïteit en differentieerbaarheid.

Opgave 1:

a.



b. $f(1) = 1$

$f(1{,}9) = 3{,}61$

$f(1{,}99) = 3{,}9601$

$f(2{,}01) = 4{,}0401$

c. voor $x = 2$ wordt de noemer nul en je kunt niet door nul delen.

Opgave 2:

a. $\lim_{x \to 1} \frac{x^2 + x - 2}{x - 2} = \frac{0}{-1} = 0$

b. $\lim_{x \to 1} \frac{x^2 + x - 2}{x - 1} = \lim_{x \to 1} \frac{(x+2)(x-1)}{x-1} = \lim_{x \to 1} x + 2 = 3$

c. $\lim_{x \to 1} \frac{x - 1}{x^2 - 1} = \lim_{x \to 1} \frac{x - 1}{(x-1)(x+1)} = \lim_{x \to 1} \frac{1}{x+1} = \frac{1}{2}$

d. $\lim_{x \to 1} \frac{x - \sqrt{x}}{x^2 - x} = \lim_{x \to 1} \frac{x - \sqrt{x}}{(x - \sqrt{x})(x + \sqrt{x})} = \lim_{x \to 1} \frac{1}{x + \sqrt{x}} = \frac{1}{2}$

Opgave 3:

a. $\lim_{x \to 0} \frac{2x^2}{3x^2} = \lim_{x \to 0} \frac{2}{3} = \frac{2}{3}$

b. $\lim_{x \to 0} \frac{2x^4}{3x^3} = \lim_{x \to 0} \frac{2x}{3} = \frac{0}{3} = 0$

c. $\lim_{x \to 0} \frac{2x^3}{3x^4} = \lim_{x \to 0} \frac{2}{3x} =$ bestaat niet

d. $\lim_{x \to 0} \frac{2x^2 - x}{3x} = \lim_{x \to 0} \frac{2x - 1}{3} = \frac{-1}{3} = -\tfrac{1}{3}$

Opgave 4:

a. $\lim_{x \to 1} \frac{2x^2 - 3x + 1}{1 - x} = \lim_{x \to 1} \frac{(1-x)(-2x+1)}{1-x} = \lim_{x \to 1} -2x + 1 = -2 + 1 = -1$

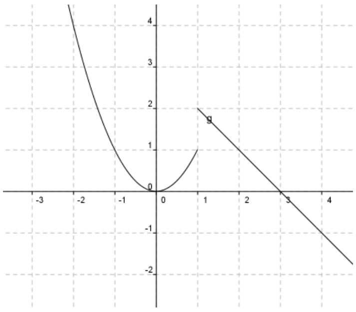
b. $\lim_{x \to 4} \frac{x - 4}{\sqrt{x} - 2} = \lim_{x \to 4} \frac{(\sqrt{x} - 2)(\sqrt{x} + 2)}{\sqrt{x} - 2} = \lim_{x \to 4} \sqrt{x} + 2 = 2 + 2 = 4$

c. $\lim\limits_{x\to 9}\dfrac{x-3\sqrt{x}}{x^2-9x}=\lim\limits_{x\to 9}\dfrac{x-3\sqrt{x}}{(x-3\sqrt{x})(x+3\sqrt{x})}=\lim\limits_{x\to 9}\dfrac{1}{x+3\sqrt{x}}=\dfrac{1}{9+9}=\dfrac{1}{18}$

d. $\lim\limits_{x\to 2}\dfrac{x^2-4}{x^3-2x^2-4x+8}=\lim\limits_{x\to 2}\dfrac{x^2-4}{(x^2-4)(x-2)}=\lim\limits_{x\to 2}\dfrac{1}{x-2}=$ bestaat niet

Opgave 5:

a.



b. nee,

c. $f(1)=1$

$-1+p=1$

$p=2$

Opgave 6:

$\lim\limits_{x\uparrow 2} f_{p,q}(x)=\lim\limits_{x\uparrow 2} x^3=8$

$\lim\limits_{x\downarrow 2} f_{p,q}(x)=\lim\limits_{x\downarrow 2} x^2+q=4+q$

$f(2)=p$

$8=4+q=p$

dus $p=8$ en $q=4$

Opgave 7:

$\lim\limits_{x\uparrow e} f_p(x)=\lim\limits_{x\uparrow e} x^2+1=e^2+1$

$\lim\limits_{x\downarrow e} f_p(x)=\lim\limits_{x\downarrow e} 3+p\ln x=3+p$

$3+p=e^2+1$

$p=e^2-2$

Opgave 8:

$\lim\limits_{x\uparrow 1} f_{p,q}(x)=\lim\limits_{x\uparrow 1} 2x-p=2-p$

$\lim\limits_{x\downarrow 1} f_{p,q}(x)=\lim\limits_{x\downarrow 1} 2^{x-1}=2^0=1$

$2-p=1$

$p=1$

$\lim\limits_{x\uparrow 3} f_{p,q}(x)=\lim\limits_{x\uparrow 3} 2^{x-1}=4$

$\lim\limits_{x\downarrow 3} f_{p,q}(x)=\lim\limits_{x\downarrow 3} x^2+q=9+q$

$9+q=4$ dus $q=-5$

Opgave 9:

a. $[-3,\rightarrow\rangle$

b. $\langle -2,\rightarrow\rangle$

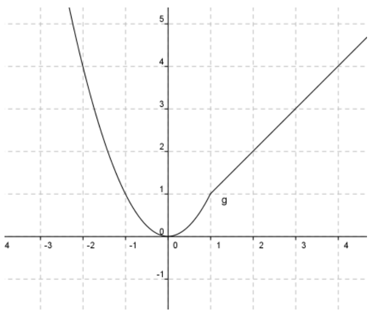
c. $9-x^2>0$
$-x^2>-9$
$x^2<9$
$-3<x<3$
$\langle -3,3\rangle$

d. $\Re$

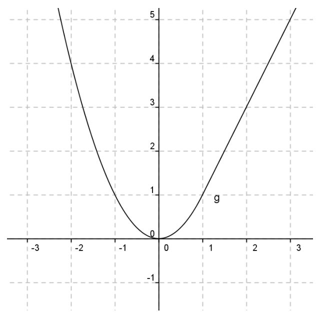
e. $\Re$

f. $2^x-4>0$
$2^x>4$
$2^x>2^2$
$x>2$
$\langle 2,\rightarrow\rangle$

Opgave 10:



a. $\lim_{x\uparrow 1} f_0(x)=\lim_{x\uparrow 1} x^2=1$
$\lim_{x\downarrow 1} f_0(x)=\lim_{x\downarrow 1} x=1$
$\lim_{x\to 1} f_0(x)=1=f(1)$
dus de functie is continu in $x=1$
ja, er zit een knik in de grafiek bij $x=1$



b. $\lim_{x\uparrow 1} g_{-1}(x)=\lim_{x\uparrow 1} x^2=1$
$\lim_{x\downarrow 1} g_{-1}(x)=\lim_{x\downarrow 1} 2x-1=2-1=1$
$\lim_{x\to 1} g_{-1}(x)=1=g(1)$
dus de functie is continu in $x=1$
nee, er zit geen knik in de grafiek

Opgave 11:

$$f(x)=|x+3|=\begin{cases} x+3 & voor\ x\geq -3 \\ -x-3 & voor\ x<-3 \end{cases}$$

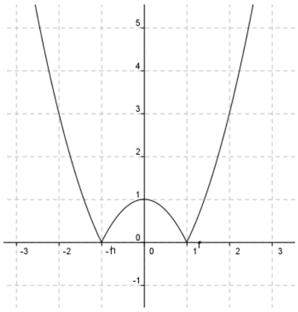
$$\lim_{h\uparrow 0}\frac{f(-3+h)-f(-3)}{h}=\lim_{h\uparrow 0}\frac{-(-3+h)-3-0}{h}=\lim_{h\uparrow 0}\frac{3-h-3}{h}=\lim_{h\uparrow 0}\frac{-h}{h}=-1$$

$$\lim_{h\downarrow 0}\frac{f(-3+h)-f(-3)}{h}=\lim_{h\downarrow 0}\frac{-3+h+3-0}{h}=\lim_{h\downarrow 0}\frac{h}{h}=1$$

De limieten zijn niet gelijk, dus de functie *f* is niet differentieerbaar in $x=-3$.

Opgave 12:

a.

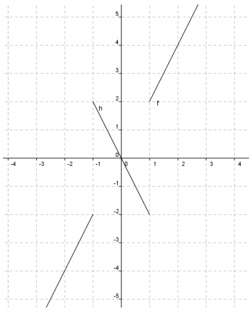


b. $|x^2 - 1| = \begin{cases} x^2 - 1 & voor\ x \le -1 \vee x \ge 1 \\ -x^2 + 1 & voor\ -1 < x < 1 \end{cases}$

$g(x) = \begin{cases} x^2 & voor\ x \le -1 \vee x \ge 1 \\ -x^2 + 2 & voor\ -1 < x < 1 \end{cases}$

$g'(x) = \begin{cases} 2x & voor\ x \le -1 \vee x \ge 1 \\ -2x & voor\ -1 < x < 1 \end{cases}$

c.



d. $x = -1$ en $x = 1$

Opgave 13:

a. $\lim\limits_{x \uparrow 2} f(x) = \lim\limits_{x \uparrow 2} x^2 - 1 = 4 - 1 = 3$

$\lim\limits_{x \downarrow 2} f(x) = \lim\limits_{x \downarrow 2} 3x - 3 = 6 - 3 = 3$

$f(2) = 3$

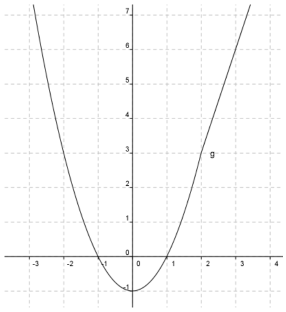
dus de functie *f* is continu in $x = 2$

b. $\lim\limits_{h \uparrow 0} \frac{f(2+h) - f(2)}{h} = \lim\limits_{h \uparrow 0} \frac{(2+h)^2 - 1 - 3}{h} = \lim\limits_{h \uparrow 0} \frac{4 + 4h + h^2 - 4}{h} = \lim\limits_{h \uparrow 0} 4 + h = 4$

$\lim\limits_{h \downarrow 0} \frac{f(2+h) - f(2)}{h} = \lim\limits_{h \downarrow 0} \frac{3(2+h) - 3 - 3}{h} = \lim\limits_{h \downarrow 0} \frac{6 + 3h - 6}{h} = \lim\limits_{h \downarrow 0} \frac{3h}{h} = 3$

f is niet differentieerbaar in $x = 2$

c.



Opgave 14:

a. $\lim_{x \uparrow 2} g(x) = \lim_{x \uparrow 2} x^2 - 2 = 4 - 2 = 2$

$\lim_{x \downarrow 2} g(x) = \lim_{x \downarrow 2} 3x - 3 = 6 - 3 = 3$

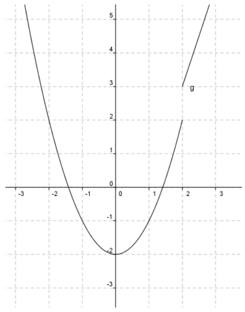
g is niet continu in $x = 2$

b. $\lim_{h \uparrow 0} \frac{g(2+h) - g(2)}{h} = \lim_{h \uparrow 0} \frac{(2+h)^2 - 2 - 2}{h} = \lim_{h \uparrow 0} \frac{4 + 4h + h^2 - 4}{h} = \lim_{h \uparrow 0} 4 + h = 4$

$\lim_{h \downarrow 0} \frac{g(2+h) - g(2)}{h} = \lim_{h \downarrow 0} \frac{3(2+h) - 3 - 2}{h} = \lim_{h \downarrow 0} \frac{6 + 3h - 5}{h} = \lim_{h \downarrow 0} \frac{1 + 3h}{h} =$ bestaat niet

dus g is niet differentieerbaar in $x = 2$

c.



Opgave 15:

a. $\lim_{x \uparrow 1} f(x) = \lim_{x \uparrow 1} -x^2 + 2x + 1 = -1 + 2 + 1 = 2$

$\lim_{x \downarrow 1} f(x) = \lim_{x \downarrow 1} x^2 - 2x + 4 = 1 - 2 + 4 = 3$

dus f is niet continu in $x = 1$

b. $f'(x) = -2x + 2$ voor $x < 1$

$f'(x) = 2x - 2$ voor $x > 1$

$\lim_{x \uparrow 1} f'(x) = \lim_{x \uparrow 1} -2x + 2 = -2 + 2 = 0$

$\lim_{x \downarrow 1} f'(x) = \lim_{x \downarrow 1} 2x - 2 = 2 - 2 = 0$

c. $\lim_{h \uparrow 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \uparrow 0} \frac{-(1+h)^2 + 2(1+h) + 1 - 2}{h} = \lim_{h \uparrow 0} \frac{-1 - 2h - h^2 + 2 + 2h + 1 - 2}{h} =$

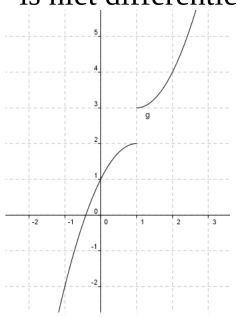
$\lim_{h \uparrow 0} \frac{-h^2}{h} = \lim_{h \uparrow 0} -h = 0$

$\lim_{h \downarrow 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \downarrow 0} \frac{(1+h)^2 - 2(1+h) + 4 - 2}{h} = \lim_{h \downarrow 0} \frac{1 + 2h + h^2 - 2 - 2h + 4 - 2}{h} =$

$\lim_{h \downarrow 0} \frac{h^2 + 1}{h} =$ bestaat niet

f is niet differentieer baar in $x = 1$

d.



Opgave 16:

a. $\lim\limits_{x\to 3} g(x) = \lim\limits_{x\to 3} \frac{1}{2}x + 1 = 1\frac{1}{2} + 1 = 2\frac{1}{2}$

$g(3) = 2$

g is niet continu in $x = 3$

b. $g'(x) = \begin{cases} \frac{1}{2} & voor\ x \neq 3 \\ 0 & voor\ x = 3 \end{cases}$

$\lim\limits_{x\uparrow 3} g'(x) = \lim\limits_{x\uparrow 3} \frac{1}{2} = \frac{1}{2}$

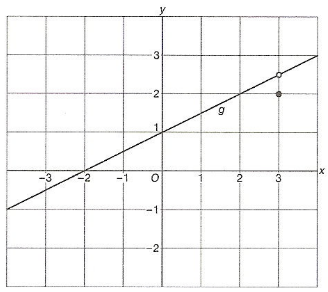
$\lim\limits_{x\downarrow 3} g'(x) = \lim\limits_{x\downarrow 3} \frac{1}{2} = \frac{1}{2}$

c. $\lim\limits_{h\uparrow 0} \frac{g(3+h) - g(3)}{h} = \lim\limits_{h\uparrow 0} \frac{\frac{1}{2}(3+h) + 1 - 2}{h} = \lim\limits_{h\uparrow 0} \frac{1\frac{1}{2} + \frac{1}{2}h + 1 - 2}{h} = \lim\limits_{h\uparrow 0} \frac{\frac{1}{2}h + \frac{1}{2}}{h} =$ bestaat niet

$\lim\limits_{h\downarrow 0} \frac{g(3+h) - g(3)}{h} = \lim\limits_{h\downarrow 0} \frac{\frac{1}{2}(3+h) + 1 - 2}{h} = \lim\limits_{h\downarrow 0} \frac{1\frac{1}{2} + \frac{1}{2}h + 1 - 2}{h} = \lim\limits_{h\downarrow 0} \frac{\frac{1}{2}h + \frac{1}{2}}{h} =$ bestaat niet

dus g is niet differentieerbaar in $x = 3$

d.



Opgave 17:

a. $\lim\limits_{x\uparrow 1} f(x) = f(1) = 1$

$\lim\limits_{x\downarrow 1} f(x) = \lim\limits_{x\downarrow 1} 3x - 2 = 3 - 2 = 1$

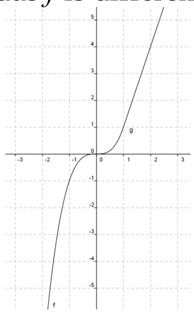
dus f is continu in $x = 1$

$\lim\limits_{x\uparrow 1} f'(x) = \lim\limits_{x\uparrow 1} 3x^2 = 3$

$\lim\limits_{x\downarrow 1} f'(x) = \lim\limits_{x\downarrow 1} 3 = 3$

dus f is differentieerbaar in $x = 1$

b.



Opgave 18:

a. $\lim\limits_{x\uparrow 0} f(x) = \lim\limits_{x\uparrow 0} f(0) = 2$

$\lim\limits_{x\downarrow 0} f(x) = \lim\limits_{x\downarrow 0} \sqrt{x+4} = 2$

dus f is continu in $x = 0$

$\lim_{x\uparrow 0} f'(x) = \lim_{x\downarrow 0} \tfrac{1}{4}x + 4 = \tfrac{1}{4}$

$\lim_{x\downarrow 0} f'(x) = \lim_{x\downarrow 0} \dfrac{1}{2\sqrt{x+4}} = \tfrac{1}{4}$

dus *f* is differentieerbaar in $x = 0$

b. $f'(x) = \begin{cases} \tfrac{1}{4}x + 4 & voor\ x \le 0 \\ \dfrac{1}{2\sqrt{x+4}} & voor\ x > 0 \end{cases}$

<u>Opgave 19:</u>

$\lim_{x\uparrow 0} f(x) = f(0) = a^2 + b$

$\lim_{x\downarrow 0} f(x) = \lim_{x\downarrow 0} x^3 + 4x + 1 = 1$

dus $a^2 + b = 1$

$\lim_{x\uparrow 0} f'(x) = \lim_{x\uparrow 0} 2(x - a) = -2a$

$\lim_{x\downarrow 0} f'(x) = \lim_{x\downarrow 0} 3x^2 + 4 = 4$

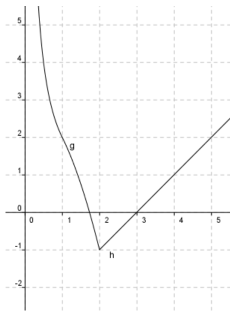
dus $-2a = 4$

$a = -2$

$b = 1 - a^2 = -3$

<u>Opgave 20:</u>

a.



b. $\lim_{x\uparrow 1} f(x) = \lim_{x\uparrow 1} \dfrac{2}{x} = 2$

$\lim_{x\downarrow 1} f(x) = f(1) = 2$

dus *f* is continu in $x = 1$

$\lim_{x\uparrow 1} f'(x) = \lim_{x\uparrow 1} -\dfrac{2}{x^2} = -2$

$\lim_{x\downarrow 1} f'(x) = \lim_{x\downarrow 1} -2x = -2$

dus *f* is differentieerbaar in $x = 1$

c. $\lim_{x\uparrow 2} f(x) = f(2) = -1$

$\lim_{x\downarrow 2} f(x) = \lim_{x\downarrow 2} px - 2p - 1 = -1$

dus *f* is continu in $x = 2$

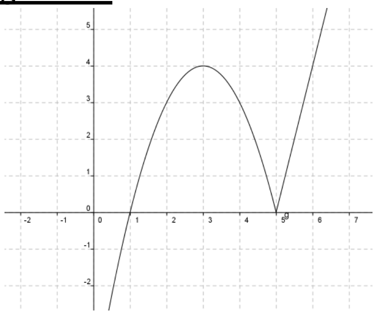
$\lim_{x\uparrow 2} f'(x) = \lim_{x\uparrow 2} -2x = -4$

$\lim_{x\downarrow 2} f'(x) = \lim_{x\downarrow 2} p = p$

dus f is differentieerbaar in $x=2$ voor $p=-4$

Opgave 21:

a.



b. $\lim\limits_{x\uparrow a} f(x) = \lim\limits_{x\uparrow a} -x^2+6x-5 = -a^2+6a-5$

$\lim\limits_{x\downarrow a} f(x) = f(a) = 0$

dus $-a^2+6a-5=0$

$a^2-6a+5=0$

$(a-1)(a-5)=0$

$a=1 \quad \vee \quad a=5$

$\lim\limits_{x\uparrow a} f'(x) = \lim\limits_{x\uparrow a} -2x+6 = -2a+6$

$\lim\limits_{x\downarrow a} f'(x) = \lim\limits_{x\downarrow a} 4 = 4$

dus $-2a+6=4$

$-2a=-2$

$a=1$

dus f_a is differentieerbaar in $x=a$ als $a=1$

Opgave 22:

a. $t(x)=x^3-8$

$t'(x)=3x^2$

$t'(2)=12$

raaklijn in $(2,0)$ is: $y=12(x-2)$

$n(x)=x^2-4$

$n'(x)=2x$

$n'(2)=4$

raaklijn in $(2,0)$ is: $y=4(x-2)$

b. $\lim\limits_{x\to 2} \dfrac{x^3-8}{x^2-4} = \lim\limits_{x\to 2} \dfrac{(x-2)(x^2+2x+4)}{(x-2)(x+2)} = \lim\limits_{x\to 2} \dfrac{x^2+2x+4}{x+2} = \dfrac{4+4+4}{4} = 3$

$\lim\limits_{x\to 2} \dfrac{12(x-2)}{4(x-2)} = \lim\limits_{x\to 2} \dfrac{12}{4} = 3$

dus $\lim\limits_{x\to 2} \dfrac{x^3-8}{x^2-4} = \lim\limits_{x\to 2} \dfrac{12(x-2)}{4(x-2)}$

$\lim\limits_{x\to 2} \dfrac{t'(x)}{n'(x)} = \lim\limits_{x\to 2} \dfrac{3x^2}{2x} = \lim\limits_{x\to 2} \tfrac{3}{2}x = 3$

Opgave 23:

a. $\lim_{x\to 3}\frac{\ln(x-2)}{x^2-9}=\lim_{x\to 3}\frac{\frac{1}{x-2}}{2x}=\frac{1}{6}$

b. $\lim_{x\to 3}\frac{x^3-27}{x^2-9}=\lim_{x\to 3}\frac{3x^2}{2x}=\frac{27}{6}=4\tfrac{1}{2}$

c. $\lim_{x\to 1}\frac{2x^3-x-1}{2(x^3-1)}=\lim_{x\to 1}\frac{6x^2-1}{6x^2}=\frac{5}{6}$

d. $\lim_{x\to 0}\frac{e^{2x}-1}{x}=\lim_{x\to 0}\frac{2e^{2x}}{1}=\frac{2}{1}=2$

Opgave 24:

a. $\lim_{x\to 2}\frac{2x^3-3x^2-12x+20}{3x^2-12x+12}=\lim_{x\to 2}\frac{6x^2-6x-12}{6x-12}=\lim_{x\to 2}\frac{12x-6}{6}=\frac{18}{6}=3$

b. $\lim_{x\to 0}\frac{4x}{\sqrt{5-x}-\sqrt{5}}=\lim_{x\to 0}\frac{4}{\frac{-1}{2\sqrt{5-x}}}=\lim_{x\to 0}-8\sqrt{5-x}=-8\sqrt{5}$

c. $\lim_{x\to 1}\frac{1-\sqrt{x}}{2-\sqrt{x+3}}=\lim_{x\to 1}\frac{\frac{-1}{2\sqrt{x}}}{\frac{-1}{2\sqrt{x+3}}}=\lim_{x\to 1}\frac{\sqrt{x+3}}{\sqrt{x}}=\frac{2}{1}=2$

d. $\lim_{x\to 4}\frac{2^x-16}{x^2-16}=\lim_{x\to 4}\frac{2^x\cdot\ln 2}{2x}=\frac{16\ln 2}{8}=2\ln 2$